THE MATHEMATICAL GAZETTE

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W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
Prof. E. T. WHITTAKER, Sc.D., F.R.S.

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VOL. XIII. OCTOBER, 1927. No. 190.

Obituary.

ROBERT FREDERICK DAVIS.

MY friendship with Davis dates from the time when we were together at King's College School and used to travel up every day to Waterloo, he from Wandsworth and I from Twickenham, and pay the toll at the turnstiles of the bridge, the dear old bridge whose fate still hangs in the balance.

Davis soon reached the top of the school in mathematics and passed on into King's College, when he worked for a year under Professor Drew, author of the text-book of the day on Geometrical Conics. Here, no doubt, Davis developed his natural gift for Pure Geometry. He gained an Entrance Scholarship at Queens' College, Cambridge, and followed me there in October 1871. He was a good musician and a keen cricketer; for cricket he had small scope at Cambridge, Queens' being then too small to have a club of its own (though he did, occasionally, organise a scratch team and arrange a few matches), but his music told in the University Musical Society and led to a close friendship with Stanford (Sir Charles Villiers Stanford), then organist of Queens'. He often devilled for Stanford on the college organ and he took his place when Stanford moved on to Trinity.

His private coach was Besant, who turned out so many brilliant mathematicians and who seems to have had the gift, not always possessed by coaches, of inspiring his pupils with a lasting love of his subject. In 1875 Davis passed out 20th wrangler, and then went to join his father in business; this was according to promise, but many of his friends have thought it was a mistake, considering his special gifts and tastes. Whether he might have gone far if he had been able to give his whole life to mathematics, or whether his brain was all the fresher when he turned to his problems with relief from less congenial work, it is impossible to say; anyhow things did not turn out well, the business began to fall off, his father put in much of his private fortune in the hope of tiding over bad times, and died.

As mathematics was not to be his profession Davis made it his hobby, and soon began to send notes and solutions to the columns of the *Educational Times*, continuing to do so until that feature of the paper was dropped. Some of his work has appeared also in the *Mathematical Gazette* and the *Notes* of the Edinburgh Mathematical Society. His ingenious mind was always attracted by a puzzle, and he spent some time and labour on the Theory of Numbers and Diophantine problems, but his special gift was Pure Geometry.

His articles and solutions put him in correspondence with many other mathematicians—Langley, Tucker, J. J. Milne, Genese, T. C. Simmons, and others. He was much attracted by the Geometry of the Triangle, its allied circles and points, and here his expert knowledge of Trilinears fitted in and rounded off the subject.

Davis joined the A.I.G.T. in 1887, served his turn on the Council of the M.A., and for many years acted as Honorary Auditor.

In 1885 Milne published *Weekly Problem Papers*, followed by *Solutions of Weekly Problem Papers*. Davis read through the proofs of the latter and made several contributions; in the preface Milne writes "my thanks are due ... especially to Mr. R. F. Davis of Queens' College, Cambridge, who supplied many neat solutions, particularly of geometrical problems ... it is a source of great regret to me that I was not able to avail myself further of his aid." When these books were followed by the *Companion to the Weekly Problem Papers*, to Davis were entrusted several sections of the work, in particular a geometrical proof of Feuerbach's Theorem, and the Rev. T. C. Simmons, who wrote a long section on the "Recent Geometry of the Triangle," says, in a letter to Milne: "The whole work was reviewed and criticised by Mr. R. F. Davis, and to his genial advice and geometrical skill I have been, in almost every chapter, indebted."

In 1890 appeared Part I. of *Geometrical Conics* under the joint authorship of Milne and Davis; Part II. followed in 1894. Of late years overwork, declining health, and other anxieties have lessened his output, but there was no falling off in his love of his subject or his mental power; almost to the last nothing gave him greater pleasure than to help a friend in some difficult problem, often solved on the back of an envelope, on his way to or from the city. All who knew him will, I think, agree with me that he was a man eminently built for friendship, with a very active brain and a most lovable disposition.

C. E. WILLIAMS.

GLEANINGS FAR AND NEAR.

449. Anagrams. Newton, as to primary ideas *not new*, but as to all else *went on*; in perceptions *no newt*; as to reputations *won ten*, that is, *ten now*, not being fully appreciated in his own time.—De Morgan to Sir W. R. Hamilton. Graves, *Life of Sir W. R. Hamilton*, iii. 554.

450. Newton's ancestry appear to have been in no way remarkable for intellectual ability, and there is nothing of note that I can find out among his descendants, except what may be inferred from the fact that the two Huttons were connected with him in some unknown way, through the maternal line.—F. Galton, *Hereditary Genius*, p. 220 (1869).

451. Anthony Sinnot, *Gent. Mag.*, July, 1800, writes that he had visited Westminster Abbey and noticed that many statues and monuments were injured. Unreproved youths were playing bat and ball among them, and throwing stones. The monument "erected over that luminary of science, Sir Isaac Newton, (had) two fingers broken from the left hand of the figure."

452. Newton remarked of antiquaries:—"I cannot imagine the utility of such studies. All their pursuits are below nature." He quoted with evident approval a saying of Barrow, that "poetry is a kind of ingenious nonsense."

In the latter connection, cf. Gray's *Ode for Music*:—

> 'Twas Milton struck the deep-ton'd shell,
> And, as the choral warblings round him swell,
> Meek Newton's self bends from his state sublime
> And nods his hoary head, and listens to the rhyme.

THE APPROACH TO THE CALCULUS.

By D. K. Picken, M.A.

Now that the Infinitesimal Calculus is becoming part of regular school courses in Mathematics, the question of how the subject should be introduced takes on correspondingly increased importance. And the question is really of fundamental importance; for it is only when the average person, as distinguished from those of special aptitude for a subject, has to be considered, that we are driven back upon the necessity for a sound common-sense basis of theory. Recent articles in the *Gazette* are evidence of the growing interest in this question—discussion of the perennial issue of e being only a part of this wider inquiry.

The writer believes that a certain fundamental change is necessary in the point of view—on the following lines:

1. *Number.*

1, 1. The first point is the necessity to face boldly the question of "Number," as being basic to mathematical analysis. We must no longer be afraid to present essential mathematical facts and ideas, because they involve certain intrinsic difficulties. And we must recognise that it is possible to *know* certain fundamental things quite accurately, without having to face at that stage all the details of *proof* associated with them. If the facts are sufficiently basic, they generally carry conviction—which permits the deferring of detailed difficulties of proof.

1, 2. The key to the idea of "Number," in the full mathematical sense, is the system of seven algebraic operations, viz. Addition and Subtraction; Multiplication and Division; Involution, Evolution and Logarithmation.* These are inherent in the elementary system of "the Natural Numbers" (one, two, three, ...), and are there simply a set of variants on the theme of the basic operation of Addition—but variants of essential significance. In the general mathematical scheme, they are developed along with the idea of Number in such a way (controlled throughout by practical requirements) that the final set of operations (reduced formally to *six* by the formal inclusion of Evolution under Involution) is applicable *without restriction* to the final system of numbers.

1, 3. The Positive and Negative Rational Numbers (of elementary theory) have the most intimate relationship with the first four operations. But the Real Numbers—from which there is no possible escape in a proper understanding of the Calculus—while they go a long way toward meeting the need of the last three operations, actually originate in the conception of Ratio, which represents the practical aspect of the development of number theory. The elementary idea of Ratio is concerned with commensurable physical quantities (more particularly with commensurable *lengths*), the ratio of two such quantities being best defined *as* a rational (ratio-n-al) number; † and the Irrational Numbers are best introduced as the numbers which meet the need for a ratio $OP : OU$ when P is collinear with given points O and U, but the lengths OU, OP are incommensurable. That such numbers "exist" is purely a question of whether they can be so defined as to combine with the Rational Numbers in the seven operations; and that is precisely what the definition by Dedekind-*schnitt* achieves. The "separation" of the Rational Numbers into two classes, for specification of an irrational number, corresponds exactly to the separation by a point, in the above line-scheme, of "the

* The writer has published a treatment of this subject in a little book, *The Number-System of Arithmetic and Algebra* (Melbourne Univ. Press; London, Macmillan & Co.), of which there is a copy in the M.A. Library: referred to below as "N.S."

† See *Gazette*, vol. x. No. 144, p. 10 (January, 1920).

rational points" of that scheme (*i.e.* of the positions of P for which the ratio $OP : OU$ is a rational number).

1, 4. The system of "Real Numbers" (Rational and Irrational) has therefore an essential characteristic of "continuity" derived directly from the geometrical continuity of the Straight Line (and closely related to the physical continuity of Motion). This is, in fact, what Dedekind's analysis of "the essence of continuity" * amounts to, although *his* emphasis was naturally rather on the distinctiveness of arithmetical continuity, in the circumstances in which he arrived at it.

1, 5. No *exact* treatment of the operations of Algebra upon the Real Numbers can ever be elementary; but the familiar facts of their expression in terms of decimal numbers can easily be shown to have the closest relation to the exact theory. This is a point at which it is sound policy to postpone a good deal of theoretical detail; the facts are plain enough.

The point to be emphasised here is that the geometrical reference, when discussing "continuous variation" of a mathematical (number) variable, is *fundamental*. The geometrical treatment of fundamental Calculus ideas is the right one, when properly applied; but it is not a proper procedure to ignore the distinction between the geometrical and the algebraical quantities—as has so often been done in the past.

2. *Continuous Variation.*

This is simply definable, for a number-variable, x, in terms of the two particular types—(1) continuous increase, (2) continuous decrease.

2, 1. *Continuous increase* of x from one given real value, a, to another, b, is such that x, in its increase from a to b (where $a < b$), takes as a value *every* real number "between" a and b. And so, again, *continuous decrease.*

2, 2. *Continuous variation*, in general, is alternation of continuous increase and continuous decrease, united at "turning-values" (cp. *motion*, backwards and forwards, on a straight line).

3. *Continuous Functions.*

3, 1. A *function* of a variable, x, is another variable, y, of which the variation *depends* on that of x in some specific way. (For elementary theory the *dependence* is generally specified by an "expression in x," embodying instructions for determination of the value of y corresponding to any given value of x; but a *given* function may have a variety of *different* expressions, which are then "identically equal"—to use the elementary, but not quite happy, term.)

3, 2. A *constant*, with respect to x, is also best defined as a "variable"—being a variable which is *not* a function of x. (It is *function* and *constant*, not *variable* and *constant*, which are in contrast.)

3, 3. A function y is said to be a "continuous function of x" in so far as it varies continuously when x varies continuously; more particularly, "continuous in the range (a, b)," where $a < b$, if it varies continuously when x *increases* continuously from a to b.

From common-sense ideas of motion it is easy to see that the graph of a continuous function, as so defined, is a geometrically-continuous locus (a *line*, or "curve"): practically an immediate consequence of the derivation of continuous variation from the Straight Line.

3, 4. It follows from the definition that we can make the change (or "difference") of y from any particular value, y_1, as small as we please, simply by taking the change (or "difference") of x, from which it results, sufficiently

* *Essays on the Theory of Numbers* (trans. Open Court Co.), especially Essay I. on "Continuity and Irrational Numbers," Sections III. and IV.

small; *i.e.* if ϵ, real and positive, be pre-assigned, as small as we please, h, also real and positive, can be determined such that

$$|y-y_1|<\epsilon, \quad \text{if only } |x-x_1|<h.$$

3, 5. The theoretically important converse, viz. that this condition is *sufficient* for continuity of the function, can be proved from the principles stated; but the proof has, of necessity, the intrinsic analytical difficulty of Real Number theory. The *fact* may, however, be taken without proof—seeing that it *is* a true converse of a practically obvious proposition: the more so, as this very proposition is, in fact, practically taken as *defining* continuity in the standard theory. (See below, § 4, where the standard order of ideas is reversed.)

3, 6. The fundamental General Theorems on Continuous Functions are:

3, 6, 1. If u, v denote functions of x* continuous in the range (a, b), then $u+v$, $u-v$, and $u\times v$ give functions of x continuous in that range; also $u\div v$, if the range includes no value of x for which $v=0$. The standard proofs of these propositions, from the fundamental analytical condition of continuity (§ 3, 5), are easy; but they may quite satisfactorily be taken without proof at this stage.

3, 6, 2. Continuity of a given function implies specific continuity of *the inverse function*. (If $y=f(x)$ and $x=F(y)$ specify one and the same relation between x and y, then $y=f(x)$ and $Y=F(x)$ are mutually inverse functions of x.)

This is an obvious fact, of which the simplest presentation is by the continuous geometrical locus which may serve as graph of either function. (Trace through to back of paper, and there interchange notation of axes.)

3, 6, 3. If u is a function of x, continuous in the range (a, b),

and y ,, ,, ,, u, ,, ,, ,, ,, of u so determined,
then y ,, ,, ,, x, ,, ,, ,, ,, (a, b).

This is an immediate consequence of the definition of § 3, 3.

3, 7. The fundamental applications of these theorems are:

3, 7, 1. From $y=x$, simplest of all cases of continuous functions, we deduce that x^2, x^3, ... specify continuous functions, in the range $(-\infty, +\infty)$; thence, again, that *integral functions* in general, for which the representative expression is the polynomial $a_0+a_1.x+a_2.x^2+\ldots+a_n.x^n$, are continuous functions in that range; also that *rational functions* in general, specified by the quotient of two polynomials, are continuous functions in ranges which include no value of x for which the denominator vanishes—in particular, $1/x$, $1/x^2$, ... for ranges not including $x=0$.

The types of real functions inverse to these have therefore specific continuity; in particular, those specified by $\sqrt[n]{x}$ and $1/\sqrt[n]{x}$ (in obvious ranges). The graphs of x^t for t rational (integral or fractional, positive or negative) should be made familiar.

3, 7, 2, 1. Continuity of functions specified by Involution and Logarithmation is reducible to that of the two fundamental (real and one-valued) types specified by (1) a^x, (2) $\log_a x$, in which a denotes a positive real constant. These two fundamental functions are monotonic continuous functions in the respective ranges (1) $(-\infty, +\infty)$, (2) $(0, +\infty)$.† The *fact* may quite soundly be assumed, in an elementary course; and the graphs for $a>1$, and for $0<a<1$, should be made familiar.

---

* Or, *either* a given number, the other being a function of x.

† This important fundamental proposition is very closely related to the definitions of a^b and $\log_a b$ in the general real case. A proof can be given which is comparatively simple. See "N.S." ch. v. and app. iv.

3, 7, 2, 2. If $y=\log_u v$, in which u, v are as in § 3, 6, 1 (but kept *positive*), we use $y=\log_a v/\log_a u$ * and reduce to § 3, 6, 3, § 3, 7, 2, 1 and § 3, 6, 1.

3, 7, 2, 3. If $y=u^v$ (u kept positive), we use $y=a^{v.\log_a u}$,* and reduce to § 3, 6, 3, § 3, 7, 2, 1 and § 3, 6, 1.

3, 7, 3. The only other fundamental particular type is that of the elementary Trigonometric Functions (or Circular Functions). From their definition as *ratios* it follows immediately that *sine* and *cosine* are continuous functions of their argument (see § 1, 3, § 2 and § 3, 3 above); whence, again, *tangent, cotangent, secant, cosecant*, from their expressions as *quotients*, are continuous functions in ranges which exclude the zero-divisor in each case (§ 3, 6, 1); and so, again, the Inverse Trigonometric Functions have specific continuity (§ 3, 6, 2). The graphs are familiar.

3, 8. These fundamental forms may then be combined in a great variety of ways to give continuous functions.

4. *Limiting Values.*

This question arises primarily out of "failure" of functional expressions, due to the cases of failure of the operations—of which the first (and basic) one is that of the zero-divisor.

4, 1. If the function y, as specified by an expression $f(x)$, is continuous in the range (a, b) except for failure of the expression $f(x)$ *at* $x=c$, the continuity may (and commonly does) determine a number, γ, "continuous with" the values of y in the neighbourhood in question, properly regarded as value of y when $x=c$.

The number γ, so arrived at, is called "the limiting-value" of the *expression* $f(x)$ for the value c of the argument, being the value to which it *continuously* "*tends*" as x *continuously tends* to c. In elementary cases it is commonly obtainable from a second expression for the same function y, *i.e.* from an "identically equal" expression, which does *not* "fail" when $x=c$. To take the simplest of examples: $y=(x^2-1)/(x+1)\equiv x-1$, for the value -1 of x.

This definition of a *limit*, by continuity, gives at once (see §§ 3, 4 and 3, 5) the standard analytical condition for a limit. (For further extension of the conception of a "limit," see § 5 below.)

4, 2. The *fundamental propositions on Limits* are, from this point of view, immediate corollaries to the corresponding propositions on Continuous Functions (§ 3, 6, 1 and § 3, 6, 3, with § 3, 7, 2). Thus, if $f(x)\to l$ and $\phi(x)\to\lambda$, when $x\to c$, then $f(x)+\phi(x)\to l+\lambda$, etc., $\dots$, $f(x)^{\phi(x)}\to l^{\lambda}$, provided we exclude certain standard cases in which l and/or $\lambda=0$ or 1.

5. *Infinity.*

5, 1. The quantities c, γ, l, λ in § 4 are understood to be ordinary numbers. The extension to "infinity" is itself a special limit conception, in the two fundamental fields of Mathematics, viz. that of the Natural Numbers (the basic "infinite sequence"), and that of Geometry—more particularly, the geometry of the Straight Line—and thence again in the Real Number system.

Thus a number-variable may "tend to infinity" in either of two ways which are of fundamental importance: (1) by taking a sequence of increasing positive integral values, which has no end, (2) by increasing continuously, beyond any value we can name. And in a functional relation between two variables, either or both may so vary.

5, 2. For continuous variation, the case of a quotient, with divisor tending to 0, is fundamental. It is easy to see † that, *if the other quantity does not*

* The general *proofs* of the Involution and Logarithmation identities are not elementary. But the *forms* are fundamental and the general theorems are always assumed in the most ordinary use of Logarithm Tables [see "N.S." §§ 8, 9, 23, 24].

† "N.S.," appendix iii., on "Nought and Infinity," p. 70.

simultaneously vanish, the quotient $\to\infty$; and, conversely, that the quotient $\to 0$ if the divisor $\to\infty$. Hence, for *continuous* variation, tendency to infinity may commonly be reduced to the more elementary case by "substitution" of the reciprocal of each variable that so tends.

5, 3, 1. But the analytical condition for limit when the *argument* tends to infinity (viz. that if ϵ be assigned, however small, X can be determined so great that $|y-\gamma|<\epsilon$ if $x>X$; or, again, if Y be assigned, however great, X determined so that $y>Y$ when $x>X$ *) has an important extension to the case of a positive integral argument.

The simplest, and basic, case is that of the Geometric Series, where

$$f(n)=1+r+r^2+\ldots+r^{n-1} \quad \text{and} \quad \to 1/(1-r) \text{ or } \to\infty$$

according as $r<1$ or $\not<1$, when $n\to\infty$, provided we *define* "limit" for this purpose in terms of the analytical condition $|f(n)-l|<\epsilon$ if $n>N$ (on a standard principle of mathematical generalisation).

5, 3, 2. And here again (as in § 3, 5) there is a converse, of great theoretical importance, which is difficult to prove, viz. that there *exists* such a limit of $f(n)$, provided that from assigned ϵ we can determine N such that

$$|f(n)-f(m)|<\epsilon$$

if only $n>m>N$. This is the basis of the Theory of Infinite Sequences—more particularly of Infinite Series.

6. *The Differential Calculus Limit.*

6, 1. This is the limit of the quotient of corresponding "differences" $(f(x)-f(x_1))/(x-x_1)$, formed (see § 3, 4) from the expression for any continuous function—the quotient expression itself specifying a continuous function of x (see § 3, 6, 1), in so far as $f(x)$ does, except for the one new feature introduced by the vanishing of the divisor when $x=x_1$. In the graph, this quotient is the "gradient" of the chord P_1P; and its *limit* is the gradient of the tangent-at-P_1 to the graph.

6, 2. It is not necessary to occupy space here with discussion of the special Calculus notation † (which will be used in following sections).

7. *The Fundamental General Theorems of Differentiation.*

These theorems are direct consequences of the fundamental Limit theorems (§ 4, 2), which we have regarded as corollaries to the corresponding theorems on Continuous Functions (§ 3, 6).

7, 1, 1. Thus, if u, v denote continuous functions of x, which have derived functions u', v', it is easy to prove that $u+v$, $u-v$, $u\times v$ give derivatives $u'+v'$, $u'-v'$, $u'.v+u.v'$, respectively; also, provided we exclude $v=0$, $u\div v$ gives derivative $(u'.v-u.v')/v^2$.

7, 1, 2. More generally, if $y=u\pm v\pm w\pm\ldots$, $y'=u'\pm v'\pm w'\pm\ldots$; and if $y=u^{\pm}v^{\pm}w^{\pm}\ldots$, $y'/y=u'/u\pm v'/v\pm w'/w\pm\ldots$. (Cf. § 8, 2, 7 below.)

7, 2. If $f(x)$, $F(x)$ denote inversely related expressions (§ 3, 6, 2), the derivative of either is obtainable if that of the other is known. For if $y=F(x)$, and therefore $x=f(y)$, since we have $f'(y)$ the limit of $\delta x/\delta y$,‡ it follows that $F'(x)=1/f'(y)=1/f'(F(x))$. (But it is easier to apply the principle to particular cases, rather than this formula.)

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* The third case of this type is $y>Y$ if $|x-c|<h$, for $y\to\infty$ when $x\to c$.

†See *Gazette*, vol. xi. No. 166, p. 387 (October, 1923), for the writer's comments on this question.

‡ For elementary theory, we have to exclude the possibility that, in § 7, 2, δy and, in § 7, 3, δu may vanish in the immediate neighbourhood of $\delta x=0$. See Carslaw, *Gazette*, vol. xii. No. 170 (May, 1924), p. 92. This consideration does not affect § 8, 1, 3.

7, 3. For the case of a *function* (y) *of a function* (u) of x (§ 3, 6, 3), we have $\delta y/\delta x=(\delta y/\delta u)\times(\delta u/\delta x)$;* whence $dy/dx=(dy/du)\,.\,(du/dx)$.

[For general cases of Involution and Logarithmation, see § 8, 2, 5 and § 8, 2, 6, 1.]

8. *Fundamental Cases.*

These follow closely the facts of § 3, 7.

8, 1, 1. If $y=x$, then $y+\delta y=x+\delta x$; $\therefore\ \delta y=\delta x$.

Hence $\delta y/\delta x=1$ and $\therefore\ dy/dx=1$.

8, 1, 2. Again, if y is *constant* (to be regarded as an *extreme* case : see § 3, 2),

then $\delta y=0$; $\therefore\ \delta y/\delta x=0$; whence, again, $dy/dx=0$.

8, 1, 3. From these two exceptionally simple cases, we may then deduce (by § 7) the differentiation of the other "algebraic" forms. In particular, if n positive integral, $y=x^n$ gives (§ 7, 1, 2) $y'/y=n/x$; and, similarly,

$$y=x^{-n}=1/x^n \text{ gives } y'/y=0/1-n/x=(-n)/x.$$

Again, if $y=x^{p/q}=\sqrt[q]{x^p}$, p, q being integral and q positive, then $y^q=x^p=u$, say ; and from $\delta y/\delta x=(\delta u/\delta x)/(\delta u/\delta y)$,* we get

$$dy/dx=(du/dx)/(du/dy)=(p\,.\,u/x)/(q\,.\,u/y)=(p/q)\,.\,(y/x)$$

—using the results just obtained for the cases of integral exponents.

Thus, for t *rational*, $y=x^t$ gives $y'=t\,.\,y/x=t\,.\,x^{t-1}$.

8, 1, 4. But it is important to observe that the same kind of method is not applicable to x^a when a is *irrational* [see § 8, 2, 6, 3 below].

8, 2, 1. If $y=\log_a x$, *a given* real and positive ; y real,

$$y+\delta y=\log_a(x+\delta x);$$

$$\therefore\ \delta y=\log_a(x+\delta x)-\log_a x=\log_a\left(1+\frac{\delta x}{x}\right),$$

and

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}\,.\,\log_a\left(1+\frac{\delta x}{x}\right)=\frac{1}{x}\,.\,\log_a(1+\xi)^{1/\xi},\ \text{if } \xi=\frac{\delta x}{x},$$

using standard Logarithmation identities *for the general real case.*†

For reality, we suppose x real and positive ; and, if we exclude $x=0$, to keep y finite, the differentiation in question is reduced to determination of the limit of $(1+x)^{1/x}$ when $x\to 0$.

8, 2, 2. This limit-problem is fundamental. It is solvable thus :

(i) $(1+x)^{1/x}$, for x varying *continuously* to o, includes, in particular, $(1+1/n)^n$ for n *positive integral*, however great ;

(ii) and the "infinite *sequence*" determined by this latter expression converges to the limit determined by $\left(1+1+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)$, *i.e.* to the limit which is the "sum" of the "infinite *series*"

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots \textit{ ad inf.},$$

a simple convergent series of which the sum is easily proved *not* rational ; the irrational number so determined is the number we denote by

$$e(=2{\cdot}7182818285\ldots).$$

(iii) The conditions for continuity and for limits (§ 3, 5, § 4 and § 5) may then be used to prove that $(1+x)^{1/x}$, which fails at $x=0$, but specifies a function otherwise continuous in the neighbourhood,‡ $\to e$, as "limiting value," when $x\to 0$.

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* See last footnote on p. 411.

† See footnote to § 3, 7, 2, 2 and § 3, 7, 2, 3.

‡ § 3, 7, 2, 3.

The *steps* indicated in this subsection are straightforward and obvious, but *the details of proof are essentially difficult*; even the easiest (by far) involves the ideas of convergency, which can hardly be regarded as elementary. This seems to be one of the clear cases where the *line* of proof should be understood —as being of essential and fundamental importance—but the actual proofs postponed.

8, 2, 3. We have then (§ 8, 2, 1 and § 4, 2)

$$\text{if } y=\log_a x, \quad y'=(\log_a e)/x\,;$$

and, therefore, in particular,

$$\text{if } y=\log_e x, \quad y'=1/x$$

—a result the extreme simplicity of which makes it of great importance in mathematical theory: an importance reflected in the fact that for $\log_e x$, in practically all theoretical work, we write simply $\log x$.

It is this specially simple result we *remember*; and from it we can at once recover the *form* of the more general result from which we obtained it, by using $\log_a x=\log x/\log a$.

8, 2, 4, 1. If, now, $y=e^x$, then $x=\log y$ and $dx/dy=1/y$; hence

$$dy/dx=y=e^x \text{ [}y \text{ here real and positive]}.$$

8, 2, 4, 2. For the more general case of $y=a^x$, we may proceed in three different ways—which are, of course, equivalent—viz.:

(1) independently, from first principles, we arrive at the same limit-problem as in § 8, 2, 1 [since $(a^{\delta x}-1)/\delta x=\xi/\log_a(1+\xi)$, if $a^{\delta x}=1+\xi$];

(2) from § 7, 2, using $x=\log_a y$ and § 8, 2, 3;

(3) from $\log y=x.\log a$, whence $y'/y=\log a$ (§ 7, 3), and

$$\therefore\ y'=y.\log a=a^x.\log a.$$

We shall see (§ 8, 2, 6) that (3) is the best in principle.

8, 2, 5. If $y=\log_u v=\log v/\log u$, u, v functions of x, supposed real and positive, we get $y'=v'/(v.\log u)-u'.\log v/(u.(\log u)^2)$.

8, 2, 6, 1. If $y=u^v$, u, v real functions of x, and u supposed positive, we have $\log y=v.\log u$; whence $y'/y=v.u'/u+v'.\log u$.

8, 2, 6, 2. This gives, in particular, § 8, 2, 4, 2, when $v=x$ and u constant.

8, 2, 6, 3. Again, it gives the case of § 8, 1, 4, when $u=x$ and v is constant (but not necessarily rational). Thus, if $y=x^a$, $\log y=a.\log x$;

$$\therefore\ y'/y=a/x \quad \text{and} \quad y'=a.y/x=a.x^{a-1}.$$

This, of course, accounts for the common form of the results of § 8, 1, 3 obtained there by much more elementary procedure.

8, 2, 7. The simple form of the general result for differentiation of a product-and-quotient expression (§ 7, 1, 2) is also accounted for, similarly. For

$$y=u^{\times}_{\div}v^{\times}_{\div}w^{\times}_{\div}\ldots \text{ gives } \log y=\log u\pm\log v\pm\log w\pm\ldots\,;$$

whence $y'/y=u'/u\pm v'/v\pm w'/w\pm\ldots$.

[*N.B.*—§ 8, 2, 5 and § 8, 2, 6, 1 complete the scheme of forms based upon the fundamental operations (see § 7 and § 1, 2).]

8, 3. In the case of the Trigonometric Functions, there are two points to be specially noted:

8, 3, 1. From the elementary point of view, these are, of course, functions of a variable *angle*; but the significance of this in the present context is obscured by the unfortunate premature use of Circular Measure in Elementary Trigonometry (*e.g.* $\cos \pi/3$ for $\cos 60°$. See § 8, 3, 2 below).

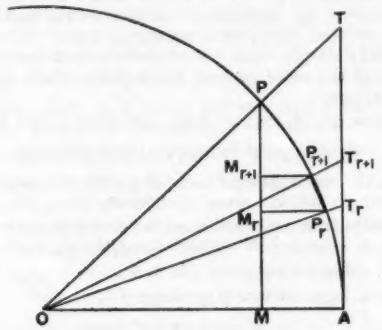
The functions being themselves *number*-quantities (the Trig. *ratios*), there can be no question of differentiation except w.r.t. an argument which is also a number-quantity (to permit the *quotient* $\delta y/\delta x$). Hence we must use for argument the *measure* of the angle in terms of *some* unit-angle.

The standard procedure then reduces the differentiation to determination of the limit of (sine/measure) for infinitesimal angle.

8, 3, 2. Elementary investigation of this limit depends fundamentally on the geometrical inequality

$$MP < \text{arc-length } AP < AT$$

for circle-sector of acute angle AOP, as in the diagram :



an inequality immediately obtainable from the elementary one

$$M_rM_{r+1} < P_rP_{r+1} < T_rT_{r+1}\ ;\quad r=0,\ 1,\ \dots\ n-1\ ;\quad P_0=A\ ;\quad P_n=P,$$

by summing, and using the limit-definition of arc-length AP.

We thus obtain

$$\sin AOP < \text{circ. meas. of } \angle AOP < \tan AOP, \text{ for acute } \angle AOP\ ;$$

and it becomes apparent that the differentiation of the functions will take its simplest form if the *number*-argument, x, used, is the *circular* measure of the angle-argument.

This is, of course, *the basic reason for the importance of circular measure to all theoretical use of these functions*—on account of which we write $\sin x$, etc., when we mean the sine, etc., of the angle whose *circular measure* is x.*

We have thus $\sin x < x < \tan x$,† if $0 < x < \pi/2$,

and, thence, $x \,.\, \cos x < \sin x < x < \tan x < x \,.\, \sec x$,

whence $\sin x/x$ and $\tan x/x \to 1$ when $x \to 0$.

The standard results of differentiation then follow, *with their essential reference to circular measure.*

8, 3, 3. For the Inverse Trigonometric (or Circular) Functions—meaning, now, *circular measure* of angle whose sine, etc., is x—the point to be noted is the infinite many-valuedness of the function in each "real" case, while

* Cp. § 8, 2, 3.

† I remember Mr. J. P. Gabbatt, then professor of Mathematics in Christchurch, N.Z., showing me a purely analytical proof of this inequality—of which the detail was strikingly close to that of the above elementary geometrical proof.

the derived function is only *one*-valued in the cases of tan, cot, only *two*-valued in the other four cases. The corresponding graphical facts—as to one, or two, sets of parallel tangents—should be emphasised. (Reduction to one-valued functions is not here a satisfactory procedure—as it is, *e.g.*, in the cases of a^x and $\log_a x$, where the "reality" does it. See "N.S." p. 39 and p. 45.)

9. This completes the Introduction to the Differential Calculus, *i.e.* the groundwork of the theory. The treatment turns on a close reference to fundamental Number theory; and, so, on a geometrical foundation for continuous variation; and on subordination of Limits to Continuity; the question of e is incidental. It has borne the test of constant university teaching for more than twenty years.

Ormond College, University of Melbourne, Sept. 1926. D. K. PICKEN.

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453. Frederick the Great found mathematics too "parching to the intellect."

454. *Beyle* (author of *Hist. de la Peinture en Italie*) suggested that "pour avoir des Newtons, il faut semer des Benjamin's Constants."

455.

Bergère, détachons nous
De Newton, de Descartes;
Ces deux espèces de fous
N'ont jamais vu le dessous
Des cartes,
Des cartes,
Des cartes.

—Saint Aulaire, in an epigram for the *Duchesse du Maine*.

456. The boys [at the school at Kendal, kept by Jonathan and John Dalton, *circa* 1785] were all glad to be taught by John Dalton, because he had a gentler disposition, and besides, his mind was so occupied with mathematics, that their faults escaped his notice.—Henry's *Memoirs of John Dalton, F.R.S.*, 1854.

457. (Sir Isaac) in Divinity and Chronology is as much below many others as he is above.—Warburton to Doddridge, Feb. 12, 1738-9.

458. Dr. Monk: "Bentley claims the undoubted merit of having in these sermons (Boyle Lectures) been the first to display the discoveries of Mr. Newton in a popular form, and to explain their irresistible force in the proof of a Deity." And Newton wrote to Bentley: "WHEN I WROTE MY TREATISE ABOUT OUR SYSTEM, I HAD AN EYE UPON SUCH PRINCIPLES AS MIGHT WORK WITH CONSIDERING MEN, FOR THE BELIEF OF A DEITY, AND NOTHING CAN REJOICE ME MORE THAN TO FIND IT USEFUL FOR THAT PURPOSE."

459. For Rohault's volume on physics used in Cambridge, and the work of Samuel Clarke, Whiston, etc., *v. Cambridge under Queen Anne*, edited by Prof. J. E. B. Mayor, pp. 279-284. In geometry the text-book was:—"De Chales, the Elements of Euclid explained in a new, but most easie method. Together with the use of every proposition through all parts of the mathematicks. Written in French by that exccellent mathematician, Claude Francis Milliet de Chales, of the Society of Jesus. And now carefully done into English, and purg'd from a multitude of errors, which had escap'd in the original. Oxford. Printed by L. Lichfield, printer to the University, for Anthony Stephens, bookseller near the theater in Oxford, 1685, 8vo." Contains bks. i-vi. xi, xii. 2nd and 4th edns. London, 1696, 1704. He was born at Chambery in 1621, taught rhetoric and classics for nine years, was for some time a missionary in Turkey; then prof. of hydrography at Marseilles; he then taught at Lyons philosophy for four years, mathematics for seven, theology for five; lastly rector of college at Chambery and died at Turin, 1678.

THE CHOICE OF UNITS IN THE TEACHING OF MECHANICS.

Discussion at the Annual Meeting of the Mathematical Association, 1927.

Mr. G. Goodwill, in opening this discussion, said that in dealing with the subject of units in mechanics, it seemed to him to be a help, in the first place, to recognize that, in general, when expressing the value of any quantity, the unit employed belonged to one of two kinds, determined by the business in hand. It would be either (*a*) a unit represented simply by a well-recognised amount of the quantity itself, or (*b*) a unit which, in the simplest possible way, took account of the relations of the quantity with others with which they might at the same time be concerned.

An illustration would make his meaning clear. A milkman was concerned with volumes—namely, with volumes of milk—and took as his unit the gallon or the pint or the quart. The district surveyor also was interested in volumes, such as the air space in rooms, but he did not choose the gallon as his unit, because he was concerned with the relations between air space and floor space and height of ceilings. A controversy between teachers as to whether they should follow the milkman or the district surveyor was too ridiculous to contemplate, but this was precisely the kind of conflict which for a generation seemed to have existed among teachers of mechanics. The admission of the justice of this reflection would put an end to the controversy.

In specialized branches of applied science in which the relations of force to other physical quantities were of subordinate importance, forces were measured in units which represented merely well-recognised forces, and they would continue to be so measured however much one party to the conflict might continue to lament.

In the same way, the draughtsman would continue to measure his angles in degrees, in spite of the fact that in much of our mathematical teaching boys were expected to assume that of course the angles were measured in radians. No architect would attempt to employ foot-pound-second units in determining the stresses in a roof frame, and those who in pursuit of uniformity attempted to restrict themselves to gravitational units were brought up against similar absurdities. He was uncertain, but he presumed they would still employ the formula $P/W = f/g$ to determine the acceleration towards its nucleus of an electron in a helium atom situated in the sun.

He could not emphasize too strongly his view of the necessity to cultivate a choice of units appropriate to the particular problem in hand. There would thus be avoided that all too frequent occurrence of a boy finding himself later on faced with problems of molecular and electrical mechanics worked out in units with which he was quite unfamiliar.

Now there was the question as to which units should be taken as fundamental, and here again there had been conflict, but it was submitted that in its philosophical aspect this conflict did not greatly concern teachers. In one aspect, however, they certainly were concerned. If it was granted that in cases in which they were dealing with a quantity in its relations with other quantities, units based on these relations should be chosen, then of a large number of such related quantities it was convenient to choose a very small number, and to express the relations of each of the others to them. This procedure set up a sort of clearing-house for the complex transactions between all the quantities concerned.

The choice of quantities to compose the small group was arbitrary, and the units in which they were measured were, in this arbitrary sense, fundamental; but the choice of quantities had not been left to the teacher—it had been made—and the fundamental units were those of length, mass, and time.

With regard to these, length and time with their units were, of course, familiar, but in the case of mass there was difficulty. The question was whether this difficulty was to be shirked; his own view was that it was fatal to shirk it. True notions of mass in its strict mechanical sense were as familiar as those of length and time. The difficulty was, of course, to avoid confusion with weight. One great help was to recognise and get a precise notion of weight as a downwardly directed force. Then what was often called weight, but which clearly presented itself as quite indifferent to direction, was not weight in the dynamical sense, but mass.

With regard to the names of units, there was no necessity to have special names for any of them except the fundamental ones. Thus in referring to momentum one might speak of so many ton-mile-hour, ton-foot-second, pound-foot-second, or centimetre-gram-second units. If they had a special partiality for the poundal, or even the tonal and the stonal, there was perhaps no serious objection.

There had been no reference in what had been said to the conflict between the adherents of the British and of the metric units, since this was not a question of mechanical units as such, but part of the larger question of the desirability of a universal decimal system.

In conclusion Mr. Goodwill said that perhaps some sort of apology ought to be made for bringing forward these extremely simple and perhaps perfectly familiar notions; his reason for speaking was that, although no considerations which were not familiar had been advanced, the points he had suggested seemed to him to have been consistently ignored, and in such a way as deterred the progress and the popularity of mechanics in the schools. He had the hope that, at any rate, they might be discussed.

Professor **J. E. A. Steggall** said that Mr. Goodwill had spoken much to the point. Nothing was more difficult, both for elementary and for advanced students, than to get an idea of dimensions in space and of dynamical units. A helpful and amusing study for young people was provided by such books as *Gulliver's Travels*, with its introduction to a world of different proportions, or a little book less familiar, and now not so easily procured, called *The Romance of Astronomy*, in which dwarfs were transported to Jupiter and giants to Ceres, the latter, of course, getting on very well, while the dwarfs on a planet ten times the diameter of the earth, though of much greater density, met with great difficulties. Such books were of value in bringing before students notions of the proper units in dynamics.

Professor **Alfred Lodge** said that he thought Professor Steggall had in mind the convenient magnitude of units rather than propriety in the units themselves. This distinction was rather an important one. It was well to remember that the mechanical engineer did not have as his fundamental units length, mass, and time. The fundamental units of the mechanical engineer were length, force, and time; to him mass was a subsidiary thing. The equation $P/W = f/g$ was a fundamental one, and boys very quickly got hold of the idea involved in it. But he thought it was important to distinguish between what a mechanical engineer wanted and what a physicist or an electrician wanted as the fundamental quantity. Weight being the pull of the earth (to speak unscientifically) on a body, the mechanical engineer certainly made weight the fundamental quantity, but to the electrical engineer and the physicist gravitation was unimportant; to them mass was the fundamental unit, and between the mechanical engineer and the electrical engineer was a great gulf fixed into which many people fell, to their own confusion and that of others. He disliked to hear of the mixing up of two things which were incompatible. It was possible certainly, with due regard to the value of each, to change from one system of units to the other, in the same way as one could change from the system of units in static electricity to those in other forms of electricity; but one must not mix up the two as though they

were simply interchangeable, and as if the only thing which differentiated them was the difference in name. Wherever in the system based on gravitation m came in, as it would in the energy of a flywheel and in other cases, m must be replaced by W/g. If mass was to be the fundamental unit, then, anywhere that W naturally came in, it must be replaced by mg. The two systems must not be mixed. It would not do to have W for one purpose in the equation and m for the other. Numerically the work must have either W all the way through or m all the way through, the equalising relation being $W=mg$. A very nice way of calculating the kinetic energy of a body moving with the velocity v was as $W \times v^2/2g$. For the beginner he thought it was easier to keep to mechanical experiments—that is to say, not to be in too great a hurry to go on to the fundamental units, those which were not confined to this earth, but pertained to the universe in general. These latter must come sooner or later, but for the beginner it was easier to start by not getting that m into the equation; the work should be continued for as long as possible with W.

Mr. **T. F. Lynam** said that according to Mr. Goodwill's remarks the fundamental units were length, time, and mass. Now, the modern development of science seemed to place a good deal of emphasis on the fact that nothing must be considered as fundamental unless it was derived from actual experience, and there was no actual or direct experience of mass at all. Many mathematicians had arrived at a concept of mass, but the original and direct concept they had was that of force. Another point was that, of course, if one reduced the units to the simplest number there remained only length and force, because the natural unit of time was length, or perhaps an angle. There were no means of measuring time directly except by length. If one took the electrostatic units and the electromagnetic units, and if it was assumed that the dimension of time was length, then it would be seen that these two sets of units were identical, and as Clerk Maxwell showed, at every point of the electromagnetic field the state of things could be defined by two vector quantities, electric force and magnetic force; from this basis Clerk Maxwell arrived at his conclusion with regard to the velocity of light. The object of teaching mechanics was to train the reasoning and investigating powers of the individual, and also to enable him to enter upon practical trades or on research work in science. In any of these cases it would be found that different units were required. The milkman—to use Mr. Goodwill's illustration—could in some ways be compared with the seller of wallpaper, for both used certain measures of length, but he did not think that the analogy between the milkman and the architect was a good one. When it came to dealing with actual things appealing to the eyes, length, the notion of time as length, and force were the most natural for the boy or any growing person to assume as fundamental units.

Mr. **G. A. Gardner** said that he did not know whether it would be pertinent to the discussion to mention the units employed in structural mechanics and statics. There was a great confusion with regard to moment of inertia. As a result of a symposium which had taken place in America twenty-six definitions of moment of inertia had been received, including "a quantity in inches," "a mathematical expression," "a length in inches," and so on. The student had great difficulty in appreciating what it really was. The speaker submitted that this question of the various units needed a great deal of revision, and as these units did appear in some of the very best text-books in constructional mechanics in such a way as to lead to confusion it was time something was done to rationalize the usage.

Dr. **W. F. Sheppard** said that it seemed to him that there was a great deal more rationalizing necessary. The question that arose first with regard to this matter was, What business was it of the teacher to choose units at all? If he was dealing with particular problems the units were stated in the problem.

One thing that had puzzled him rather in the discussion so far was what people meant by "fundamental." Certain quantities might be "fundamental" psychologically, certain others might be "fundamental" scientifically, and these again were going to be subdivided apparently into those fundamental for ordinary use on this earth, and those which related to astronomy or which might possibly have to be used by the inhabitants of Mars. Then there had been a reference to "fundamental in the philosophic sense." He had no idea what this meant. Mass was obviously fundamental. When he bought a pound of butter he had an idea as to a certain quantity of butter he expected to receive. The attraction of the earth might have been doubled during the night, and he might know nothing about it, but if the mass of butter which the dairyman put on the scales was not what he expected as the result of experience a pound to be it would not satisfy him. With regard to the choice of units in teaching, the question was whether one was going to keep his units in his work or not. He had looked over a good many examination papers which dealt with the teaching of mechanics, and he found the popular method was to say that one was using such and such units and then to do everything by number. The result was that answers were constantly coming out wrong, and the most frequent reason was that they were given in the wrong units. If the units had been kept all the time instead of being expressed only by number, the candidate would have seen that that was so. Momentum was mixed up with work done, and so forth. When one saw a numerical result which was wrong one could tell how the multiplication of it by certain omitted factors would bring it to the right dimensions.

Mr. **G. A. Gardner** asked whether it was a common method to express the necessary categories in algebraical form, simply by letters, and add the unit to them. He did not follow that method altogether as a general rule, but it overcame a lot of trouble by giving the student an expression. The student was not told what it actually was, whether inches or pounds, but he should be able by working it out for himself to get clear ideas on the subject.

Professor **Lodge** (from the Chair) said that the meeting had not really thrashed out all the points covered by the introductory paper, but the scheduled time had more than arrived for the next item, and they could only express their thanks to Mr. Goodwill for opening the discussion.

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460. Bullialdus (Bouillaud)... had the good fortune to make a guess, which, had he been Newton, would not have lain idle in his hands. He asserts in opposition to Kepler, that the law of the attracting force of the sun, if such a thing there be, cannot be inversely as the distances, but *inversely as the squares of the distances.* He is thus the first who started this notion.—De Morgan, *Penny Cyclopaedia.*

461. The family of Tycho Brahe "was as noble and as ignorant as sixteen undisputed quarterings could make them."—De Morgan, *Penny Cyclopaedia.*

462. The only saying preserved of Henry Briggs is that he once called astrology "a mere system of groundless conceits."—De Morgan, *Penny Cyclopaedia.*

463. Some persons say that Dante meant theology
By Beatrice....
I think that Dante's more abstruse ecstatics
Meant to personify the mathematics.
—Byron, *Don Juan*, iii. 11.

464. Harriet Martineau's translation of *The Philosophy of Comte* states that:—"O Physics, beware of Metaphysics!" was a favourite saying of Newton.

MATHEMATICAL NOTES.

874. [K. 2. a.] *The envelope of Simson's line.*

Let H be the orthocentre of $\triangle ABC$, and P, P' two adjacent points on the circumcircle whose pedal lines meet at V. Then it is well known that R, R', the middle points of HP, HP' respectively, lie on the 9-point circle.

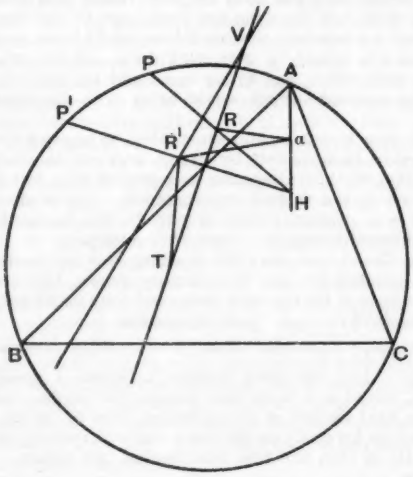
Let α be the mid-point of HA and let VR cut the 9-point circle again at T. Now the angle between the pedal lines of P and P', viz.

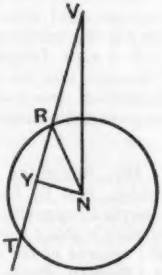
$$\angle RVR' = \angle PAP'$$
$$= \angle R\alpha R' \quad \text{since } PA \text{ is parl. to } R\alpha \text{ and } P'A \text{ to } R'\alpha$$
$$= \angle RTR' \quad \text{since the 4 points lie on 9-point circle;}$$
$$\therefore\ R'V = R'T.$$

Hence when P' moves up into coincidence with P, R' and R coalesce, and

$$TR = RV$$

where V is the point at which the pedal line TR touches its envelope.





Let N be the 9-point centre, and draw NY perpendicular to the pedal line TR.

Let $NV = r$, $NY = p$, $NR = a = \left(\frac{R}{2}\right)$.

Then
$$NV^2 - NR^2 = VY^2 - RY^2$$
$$= (3RY)^2 - RY^2$$
$$= 8\,(NR^2 - NY^2),$$

so
$$r^2 + 8p^2 = 9a^2,$$

which is the p, r equation of a three-cusped hypocycloid, N being its centre, $\frac{R}{2}$ the radius of the rolling circle and $\frac{3R}{2}$ the radius of the fixed circle.

R. F. D.
E. P. L.

[Extract from Mr. Lewis's letter to the Editor on the work of the late Mr. R. F. Davis:

"Once, some years ago now, I sent him my deliberations on the pedal line, and he returned my manuscript with one or two elegant improvements. The best tribute I can pay to his memory is to send you the enclosed. The first half is his work. I had proved $TR=RV$ in a cumbersome way, and he produced what you see, almost by return of post."]

875. [V. 8.]

Prof. Gino Loria asks for particulars of the life of William Braikenridge (or Brakenridge), F.R.S. He was Rector of St. Michael's, Bassishaw, London, and Master of Sion College Library. His death is noticed in the *Gentleman's Magazine*, July 30, 1762. He is mentioned six times in Chasles' *Aperçu*. In the British Museum are eighteen letters from Braikenridge to Birch. His *Exercitatio Geometria de descriptione Linearum Curvarum* appeared in 1733, and, according to Prof. Loria, in the *Phil. Trans.* for 1735.

Further details will be welcomed.

876. [J. 2. a.] A, B and C take part in a three-cornered running race in which a tie between any two (or all three) is impossible, *i.e.* of negligible probability.

(*a*) Given A's chance of beating B and A's chance of beating C, what is A's chance of beating both, *i.e.* of winning?

(*b*) Given A's chance of beating B and B's chance of beating C, what is A's chance of beating C? S. GOODING.

877. [L¹. 7; T. 3. a.] *Problem. Solution required.*

Will some reader kindly send to the *Gazette* a solution of the following problem?

L is a point of light situate at (x_1, y_1, z_1). A metal disc of radius a and of negligible thickness whose centre is at (x_2, y_2, z_2) lies in the plane

$$l_1x+m_1y+n_1z=p_1.$$

Find the foci and eccentricities of the conic sections formed (i) by the reflection of light from L upon the plane $l_2x+m_2y+n_2z=p_2$, and (ii) by the shadow of the disc upon the same plane. "ELPHIN."

878. [L¹. 1.] *Elements at Infinity.*

It is sometimes desirable to introduce the point at infinity P_∞ on a straight line AB by a metrical definition, illustrating it by considering the values of the ratio AP/PB as P moves along the straight line AB. In this case the following sequence of theorems is suggested:

(1) An expression in which P_∞ occurs as the end-point of a segment means the limit of the expression in P as $P\to\infty$ on the straight line. Hence, by assuming the theorem on the limit of a rational function, we postulate the existence of an ideal point P_∞ on the straight line AB, such that

$$AP_\infty/P_\infty B=-1,$$

where A, B are two finite points. P_∞ may be used as the end-point of a segment in a rational function of segments, subject to the conditions

$$AP_\infty+P_\infty B=AB$$

(where A, B are any finite pair of points on the straight line) and

$$AP_\infty/P_\infty B=-1,$$

provided the expression is determinate and finite. Thus $AB/BP_\infty=0$.

(2) The definition is independent of the particular pair of finite points chosen on the straight line; for if C, D are any other points on the straight line, then

$$\begin{aligned} CP_\infty/P_\infty D &= \{(CA+AP_\infty)/P_\infty B\} \,.\, \{P_\infty B/(P_\infty B+BD)\} \\ &= \{CA/P_\infty B + AP_\infty/P_\infty B\}/\{1+BD/P_\infty B\} \\ &= -1. \end{aligned}$$

(3) It follows that the cross-ratio of the four points A, B, C, P_∞, where C is the conjugate of P_∞, is AC/BC, using Cremona's definition.

(4) If two straight lines AB, $A'B'$ meet at a finite point P and two parallel straight lines AA', BB' meet them in A, B and A', B', then the point of intersection is the point in AB such that $AP/PB = -AA'/BB'$. When AB, $A'B'$ are parallel $AA'=BB'$, and the straight lines may thus be considered to meet at P, where $AP/PB=-1$, *i.e.* P is the point at infinity on AB. Similarly it is the point at infinity on $A'B'$.

(5) With the same notation as in (4) it follows that when P is a finite point $AP/A'P=BP/B'P=AB/A'B'$ when AB, $A'B'$ are parallel $AB=A'B'$; hence we have in the limit, if A, A' are two points each on one of two parallel straight lines meeting at P_∞, then $AP_\infty/A'P_\infty=1$.

The remaining properties of the elements at infinity can be deduced from (4) in the usual way.

A useful graphical way of studying the variation of AP/PB along the straight line is to take A as origin, AB as x-axis, $AP=x$, $AP/PB=y$, and plot the graph $y(a-x)=x$, where $AB=a$. The value of the ratio at any point of the line itself (the x-axis) is then pictorially evident. H. V. MALLISON.

879. [L¹. 1. b.] *A Fallacy in Geometrical Conics.*

Let Ω be a point at infinity on a conic, let O_1, O_2 be any two accessible points in the plane, let two parallel lines through O_1, O_2 cut the conic in P_1, Q_1 and P_2, Q_2, and let $O_1\Omega$, $O_2\Omega$ cut the conic again in R_1, R_2. Then

$$O_1P_1 \,.\, O_1Q_1 : O_2P_2 \,.\, O_2Q_2 = O_1R_1 \,.\, O_1\Omega : O_2R_2 \,.\, O_2\Omega = O_1R_1 : O_2R_2$$

by the usual argument. But the result is wrong.

I wonder whether in ninety-nine out of a hundred applications of the method its validity is really examined more carefully than in this example. E. H. N.

880. [M. 6. Bα.] *The Pure Geometry of the Lemniscate.*

[The following note is an implicit criticism of two weaknesses in much of our teaching. No curves except the conics acquire any coherent individuality for us; the folium and the cycloid are valuable material for exercises in differentiation and integration, but to few of us are they anything more. For this very reason, when we wish to illustrate methods of geometrical transformation we hesitate to do more than derive a property of a conic from a property of a circle or connect one property of a circle with another. Inversion is a metrical transformation, and we obtain a curve rich in simple metrical properties if we apply the transformation to a curve which is itself rich in them. Of conics, perhaps the richest is the rectangular hyperbola, and the note deals with the lemniscate $r^2=a^2\cos 2\theta$, which is obviously the inverse of the hyperbola $r^2\cos 2\theta=a^2$ with respect to a concentric circle of radius a. Even the teacher who is dubious of allowing time for excursions beyond the field of conics may find that the search for transformable results is a profitable stimulus to students on whom the exercise of finding special metrical forms of general projective theorems has begun to pall.—E. H. N.]

The subject of this note is a curve which has a double connection with the rectangular hyperbola. In order to use capital letters in constructions on the described curve, we use small letters in constructions on the hyperbola.

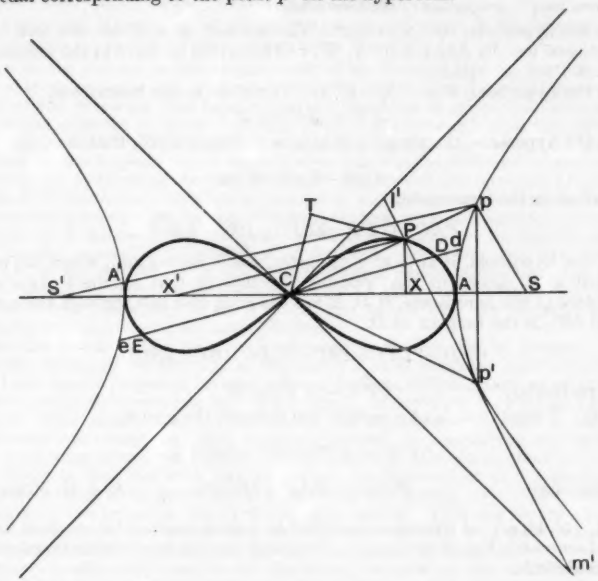
The pedal of a rectangular hyperbola with respect to its centre is also the inverse of the hyperbola with respect to its auxiliary circle.

Let p be any point of the hyperbola, and p' be the image of p in the focal axis CA; let the tangent at p' cut the asymptotes in l', m' and cut Cp in P.

Then firstly, $p'm'C=m'Cp'$, since p' is the midpoint of the base of the right-angled triangle $Cl'm'$, and

$$m'Cp'=pCl',$$

by symmetry; hence the angle from $m'p'$ to Cp is equal to that from $m'C$ to Cl'. That is to say, Cp is perpendicular to $m'p'$, or, in other words, the locus of P is the pedal of the hyperbola with respect to C, the point P on the pedal corresponding to the point p' on the hyperbola.



Secondly, because $Cp=Cp'=\frac{1}{2}l'm'$, and CP is perpendicular to $l'm'$, the product $CP \cdot Cp$ is the area of the triangle $Cl'm'$, which is a^2: the locus of P is the inverse of the hyperbola with respect to the auxiliary circle, the point P on the inverse corresponding to the point p on the hyperbola.

The theorem that the pedal and the inverse are identical may be expressed in the form that a rectangular hyperbola is its own polar reciprocal with respect to its auxiliary circle, but an enunciation which refers explicitly to the derived curve is a more natural introduction to this curve, which is known as the Lemniscate of Bernoulli.

The lemniscate is a quartic curve.

For a circle through C meets the hyperbola in four points, and by inversion any line meets the lemniscate in four points.

The direction of the tangent at P is seen most readily if the lemniscate is regarded as a pedal. If PT is the tangent and CT is the perpendicular from C, the triangle CPT is similar to the triangle $Cp'P$. Hence

$$T\hat{P}C = P\hat{p}'C = 90° - 2A\hat{C}P,$$

$$P\hat{C}T = p'\hat{C}P = 2A\hat{C}P, \quad A\hat{C}T = 3A\hat{C}P.$$

Also $$CT = CP^2/Cp' = CP^3/CA^2.$$

Polar equations.

Since the polar equation of the hyperbola is $r^2 \cos 2\theta = a^2$, that of the

lemniscate is $r^2=a^2\cos 2\theta$. The expressions found for CT and the angle ACT show that the polar equation of the locus of T, the pedal of the lemniscate with respect to C, is $r^{\frac{2}{3}}=a^{\frac{2}{3}}\cos\frac{2}{3}\theta$.

Some simple properties of the lemniscate.

In the hyperbola, $Sp\,.\,S'p=Cp^2$. The inverses of S, S' are the feet of the directrices, usually denoted by X, X', and inverting we have in the lemniscate, $XP\,.\,X'P=CX^2=\frac{1}{2}CA^2$.

In the hyperbola, $S'p \sim Sp=2CA$. Therefore in the lemniscate,

$$X'P \sim XP=\sqrt{2}CP.$$

In the hyperbola, the tangent pt bisects the angle SpS', that is,

$$\hat{CpS}-\hat{S'pC}=2\hat{Cpt}.$$

Therefore in the lemniscate,

$$P\hat{X}C-C\hat{X}'P=2T\hat{P}C=180^\circ-4A\hat{C}P.$$

In the hyperbola, if de is any diameter, $\hat{pde} \sim \hat{dep}=t'_d\,dC$, where dt'_d is the tangent at d, drawn in the opposite direction to that assumed previously. Therefore in the lemniscate, if D, E are the ends of a line through the centre, and if DT_D is the tangent at D,

$$C\hat{P}D \sim E\hat{P}C=180^\circ-T_D\hat{D}C=90^\circ+2A\hat{C}D.$$

In particular, $$C\hat{P}A \sim A'\hat{P}C=90^\circ.$$

If D', E' are the ends of a second line through the centre,

$$C\hat{P}D' \sim E'\hat{P}C=90^\circ+2A\hat{C}D',$$

and therefore $$D\hat{P}D' \sim E\hat{P}E'=2D\hat{C}D'.$$ A. J. H. MORRELL.

881. [L¹. 17. e.] *A triangle is inscribed in a fixed conic and its centroid lies on a fixed concentric homothetic conic. Prove that its nine-points circle is orthogonal to a fixed circle.*

Project the conic orthogonally into a circle, centre K. Let ABC be the projected triangle, G its centroid and O its orthocentre. Let AO cut BC in D, and the circle in H. Let P, Q, R be the middle points of BC, CA, AB respectively, and let PK cut QR in M. Then in the projected figure the locus of G is a concentric circle, and therefore KO, which is $3KG$, is constant. If ρ be the circumradius of ABC, $\rho^2-KO^2=AO\,.\,OH=2AO\,.\,OD=8PK,\ KM$, since the triangles PQR, ABC have G as centre of perspective. Hence $PK\,.\,KM$ is constant, and PQR is self-conjugate with respect to a fixed circle, centre K.

Hence in the original figure the triangle formed by the mid-points of the sides is self-conjugate with respect to a fixed concentric homothetic conic, and therefore by Gaskin's theorem its circumcircle, which is the nine-points circle of the original triangle, is orthogonal to a fixed circle, viz. the director circle of this latter conic. E. H. SMART.

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465. Porson's parody of a Cambridge Examiner, in which the Circle of the Sciences is invoked to explain the nature of a salt-box, is minutely accurate and irresistibly ludicrous. Rabelais' *Court of Queen Whim*, and Swift's *Academy of Lagado* are similar examples, but the best are Arbuthnot's "Dissertation on Toys" and his "Illustrations of Logic" in *Martinus Scriblerus*.

466. De Witt . . . had a great clearness of apprehension, a large compass of mathematical knowledge, and had applied algebra to matters of trade more than any man before him.—Bishop Burnet's *History of His Own Times*.

REVIEWS.

A Course of Differential Geometry. By the late J. E. CAMPBELL. Pp. xvi + 262. 21s. 1926. (Oxford Univ. Press.)

Differential Geometry of Three Dimensions. By C. E. WEATHERBURN. Pp. xii + 268. 12s. 6d. 1926. (Cambridge Univ. Press.)

One of the results of the recent work in the theory of relativity has been that a fresh impetus has been given to the study of differential geometry. Many of the problems that have arisen are problems in differential geometry, and in the solution of these there has grown up a new calculus, the calculus of tensors, which has had a remarkable success in this connection. This has resulted in the appearance of several new volumes on differential geometry, in which the methods of the standard works on the subject have been replaced by those which have proved so successful in satisfying modern requirements. In expressing an opinion on any of these volumes, we must consider primarily the success or failure of the use made of these new methods, for therein generally lies the reason for the existence of the book. If in this way new light is thrown on the subject, or if the presentation is made more clear, the book has achieved its purpose, and is a desirable addition to the literature of the subject; but unless this is so, it is difficult to see why the classical works of Darboux and Bianchi are not sufficient.

Mr. Campbell, in his volume, makes an extensive use of tensors. The author's death, as we are told in the editor's note, occurred before the book had been finally prepared for the printer, but the difficult task of seeing the book through the press has been successfully undertaken by Professor E. B. Elliott. The latter has, wisely, made no alterations beyond what was absolutely necessary, and has confined himself to numbering the sections and supplying some references, besides making any corrections required. The result is that the book has, in places, the appearance of a collection of separate monographs, particularly in the chapters dealing with those problems which were the principal topics of the older works. This has its advantages, but it occasionally makes reading difficult, and we wish that more references had been given when results are quoted from previous chapters.

After a preliminary chapter on the theory of tensors, the volume may be divided roughly into three sections. Three chapters are devoted to the consideration of a surface as a two-way space, and to the problem of identifying the two-way space with a locus in three dimensions, the basis on which the whole discussion is founded being the element of length. In connection with the subject to which a later portion of the book is devoted, this section is interesting as a consideration of a particular case of the more general problem, but in itself it is of importance, as it gives us a new view of certain questions which have been discussed at length in older works. The calculus of tensors is very useful here, and simplifies work considerably. The chapters which follow are devoted to a discussion of subjects such as linear congruences, minimal surfaces, triply-orthogonal systems, and so on. It is in these chapters, devoted to problems discussed at length in earlier books, that a reader becomes most critical of the new methods. The accounts of the various subjects are interesting and clear, but we have a feeling that this is due largely to the unusual insight of the author, and that it owes little, if anything, to the substitution of one method of consideration for another. And, indeed, we must admit that there is little to be gained from the use of tensors and vectors, save a shortening of writing, and this is often more apparent than real. But it is pleasing to see how the methods of the tensor calculus may be applied to these problems, and the author has succeeded in putting into very little space most of the essential points of the theory. The volume concludes with four chapters on the differential geometry of an n-way space, and it is here that any doubts as to the success of the tensor calculus disappear. Naturally, the topics of most importance in relativity are given chief place, and the problem of finding an Einstein space to contain a given n-way space is discussed at considerable length. A chapter of exceptional interest is that on spaces whose Riemann curvature is constant. Unfortunately, this section

of the book has the appearance of having been left in an unfinished state, and we must regret that the author did not live to add a chapter on the physics of Einstein, which was to complete the book. One minor point may be mentioned; contrary to the usual custom, Mr. Campbell uses the word "surface" for an n-way space, instead of restricting it to mean a two-dimensional manifold.

The volume is not suitable to a beginner in differential geometry, but it requires only a limited previous knowledge to read it and to appreciate it. A useful feature is the habit of recapitulating after a series of results has been obtained, but we might desire more references than Professor Elliott was able to recover, and a bibliography would have been valuable. Misprints are few in number and unimportant. As, however, the book is best read as a whole, it is unnecessary to discuss minute details of treatment. The volume throughout bears the mark of the author's individuality, and is one which can be recommended to all students of differential geometry.

In comparison with Mr. Campbell's work, Professor Weatherburn's volume is of a very elementary character. Great use is made of vector methods, and in this respect the author rather surprisingly claims some originality of treatment. These methods have however been superseded by the use of tensors, which have the advantage of generality, and are not open to the criticism that they do not lead easily to new results. This criticism of vector methods is one with which Professor Weatherburn disagrees, for he adds a special chapter in order to explain results which he has obtained by these means. But the fact remains that Mr. Campbell's work includes all that Professor Weatherburn's contains and more, and anyone who reads the former will not find much which is new to him in the latter.

As a textbook for elementary students, Professor Weatherburn's volume is marred by a certain looseness of expression. We soon get accustomed to the glib manner in which the author speaks of "consecutive points" and "consecutive generators of a ruled surface", but even this does not prevent us from being startled on reading, on page 140, in connection with ruled surfaces, that "as the point of contact moves along a generator from one end to the other, the tangent plane turns through an angle of 180°". There is, however, a large collection of examples, many of which are useful. W. V. D. HODGE.

Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert. Teil I. By FELIX KLEIN. Pp. xiii+385. R.M. 21. 1926. (Springer: Die Grundlehren der mathematischen Wissenschaften. Bd. XXIV.)

Klein let himself go in these lectures, and the result is a fascinating volume which, as he remarks of Monge's *Application de l'Analyse à la Géométrie*, "liest sich wie ein Roman". Not only does he show an astounding knowledge of all branches of mathematics, both pure and applied, but he is able to place the contributions of the various mathematicians of the century in perspective, and, in a few brilliant sentences, to present the personalities of the men with whom he has to deal.

Gauss is of course his hero; he is "mathematicorum princeps", he alone of his century is to be placed with Archimedes and Newton in the "Hobbs class". And so the first chapter, of some sixty pages, is simply headed "Gauss". Gauss hated teaching, and in the polished, classical form of his published works belongs rather to the eighteenth century than to the nineteenth, but the manuscripts which he left behind, and, in particular, his *Tagebuch*, show that combination of wonderful many-sidedness with detailed investigation in special directions, that complete equilibrium between inventive genius, feeling for rigour, and practical sense of applications, which make him unrivalled.

It is typical of this book that mention of "rigour" leads Klein into a digression. He remarks that in times of greatest advance, such, for instance, as the invention of the calculus in the eighteenth century, the desire for rigour remains in the background; then comes a period of criticism, when the territory acquired is consolidated and made secure. And he has a word to say in defence of scholastic philosophy. The question whether God could have created an infinite universe in an hour is much the same as the problem of infinite sets of points in the unit segment; and, in fact, Georg Cantor, the

founder of the theory of aggregates, and his predecessor, Bolzano, were brought up in the scholastic tradition.

Another digression is on politics and mathematics. Klein finds no correlation between a taste for mathematics and any particular school of political thought. Cauchy was an aristocratic conservative, D'Alembert a revolutionary, Monge a Jacobin, Jacobi a radical, Hermite belonged to the clerical party. Nor, again, is there any distinction by creed. Weierstrass was a Catholic, Jacobi a Jew, Faraday and Riemann had a somewhat naïve Protestant piety, which was quite unaffected by their intellectual development, Salmon was dogmatically Protestant.

But there is no end to the curious and entertaining scraps of information which one can pick up from this book. One knows the work of Poncelet, Möbius, and Dirichlet, but I, at any rate, did not know before that Poncelet was a representative of France at the 1851 Exhibition, that Möbius's father was a dancing-master and that his son has written a book on the feeble-mindedness of women, and that Dirichlet married a sister of Mendelssohn, who filled the house with musicians and other artistic folk, much to his disgust.

Then Klein has wise and penetrating remarks upon such matters as the curiously varied attitude of mathematicians towards the Theory of Numbers, and the point of view of physicists with regard to Weierstrass's criticism on the "proof" of Dirichlet's Principle and his continuous functions without differential coefficients. He tells a delightful story of Gordan, who by complicated algebra had managed to prove the finiteness theorem for invariants of binary forms. Twenty years later Hilbert produced his wonderful proof, very simple and very abstract, which does the thing for any number of variables and any number of forms at one go. "Das ist nicht Mathematik, das ist Theologie," said Gordan. And finally, to come nearer home, there is Salmon's remark, when Klein enquired why at Dublin there were still only three faculties, theology, classics and mathematics, whereas at Cambridge all the newest subjects were taught—"Yes, but Cambridge is so ambitious!"

I have perhaps devoted too much space to the more frivolous parts of Klein's book, but it is impossible to give an idea of the enormous extent of ground covered and the illuminating way in which the main lines of progress are traced. And when one remembers the immense amount of creative work which Klein himself did, one's admiration grows. On p. 327 he is mentioning the tendency, not merely to divide mathematics into separate chapters, but to subdivide into schools according to the method of treatment, different schools not understanding one another's language—he instances the two separate reports on functions of two variables in the *Encyklopädie*, the geometrical (by Castelnuovo and Enriques), the arithmetical (by Jung). Such a tendency, carried to its logical conclusion, would mean the death of mathematics. And then he writes his own epitaph: "Wir selbst haben immer das Umgekehrte angestrebt. In unserer Generation haben wir 1. Invariantentheorie, 2. Gleichungstheorie, 3. Funktionentheorie, 4. Geometrie und 5. Zahlentheorie mehr oder weniger in Kontakt gehalten, und das war unser besonderer Stolz."

F. P. W.

The Theory of Integration. By L. C. Young. Cambridge Tracts, No. 21. Pp. vii + 52. 5s. 1917. (Cam. Univ. Press.)

It is pleasant to welcome a new tract in the valuable Cambridge series, to which no addition had been made since 1920.

Mr. Young gives a clear account of his father's theory of integration based on the method of monotone sequences. By this method it is possible at an early stage to define a general integral and to establish some of its properties before sets of points have been studied. But the most interesting properties of the integral—in particular, the fundamental property of differentiability—cannot be approached without a knowledge of sets of points. The more usual order, which is adopted for example in Hobson's *Theory of Functions of a Real Variable*, of developing the theory of integration from that of sets of points appears the more natural.

In the first part of the tract Mr. Young explains concisely how, starting from simple functions, one may generate by the method of monotone sequences

functions which behave as badly * as one wishes. He then devotes two chapters to the theory of integration, giving a very general definition of the integral and such properties of it as do not depend on sets of points. An appendix contains sections on continuity and uniform convergence and on sets of points. The author does a service to analysis by stressing the important concept of uniformity of convergence at a point, which is not yet sufficiently appreciated.

The proof given of Theorem IV., page 37, seems inadequate. If it is not assumed that $|f|<M$ there is no reason why the second integral on the right-hand side of the first equation on page 38 should be less than M/m. The detail of the proof of the theorem on page 45 is inaccurate, though it is clear enough what is meant. We regret that Mr. Young has not adopted the accepted notation of ϵ for a small positive number—e is worked hard enough as the base of Napierian logarithms.

Mr. Young has succeeded conspicuously in his expressed endeavour to do justice to his father's ideas and to introduce some improvements of his own.

The Elements of the Theory of Real Functions. By J. E. Littlewood. 2nd Edit. Pp. viii+60. 4s. 6d. 1926. (Heffer.)

This is a revised edition of the book which Mr. Littlewood published in 1925 under the title *Lecture Notes on the Elements of the Theory of Real Functions*, primarily for the use of his pupils attending his lectures.

The present edition is slightly expanded, and the subject-matter is presented so as to be intelligible independently of the exposition given in the lecture room. To a reader not attending the lectures the book is severe. An account in sixty pages of Classes and Cardinals, Well-ordered and other Series, together with the elements of the Theory of Sets of Points, cannot be light reading. But for any one who has read Mr. Russell's admirable *Introduction to Mathematical Philosophy* and who wants to follow up the ideas in it, Mr. Littlewood's more formal account is an excellent sequel.

It is only Mr. Littlewood's disregard (until 1925) of his duty to English students of mathematics of writing books which may make it pardonable to say explicitly that his exposition is exact and clear, and that as an authoritative statement of the foundations of the theory of real functions his book has no rival.

May we suggest to Mr. Littlewood that one book is an unsatisfying contribution from him to mathematical literature, and we hope it is the first of many. Perhaps he will agree that for each 500 pages of work that he contributes to periodicals he will write a book that will appeal to a larger circle of readers. Or perhaps he will promise that when he makes another epigram as good as the one at the foot of page 2,

" " one-one " 's are not " " one "-" one " " 's ! ",

he will write a book round it.

J. C. Burkill.

The Mechanics of the Atom. By Max Born. Translated by J. W. Fisher, and revised by D. R. Hartree. Pp. xvi+317. 18s. 1927. (International Text-Books of Exact Science ; Bell.)

It is a great pleasure to welcome this translation of Prof. Born's well-known book. It is only a pity that it did not appear earlier, but this is to offer grudging thanks to those who have devoted much time and energy to the task of translation and revision. We may be glad that now we have the book in English, for every serious student of atomic theory must read it. Anyone thoroughly acquainted with Andrade's *Structure of the Atom*, who wants to know more about the underlying formal theories should turn first to the book under review.

This is not the time or place for a detailed review of Prof. Born's book, for in German it has been available now for nearly three years, and the translation differs from the original only on minor points where new facts have become known since the German original was written. But some comment may properly be made on the position of the book in view of the vehement

* Certain functions are called " well-behaved " by N. Wiener, *Mathematische Zeitschrift*, vol. 24.

advance which has been made in the theory since it first appeared. This point is dealt with by Prof. Born in a special preface to the English edition. He finds, he says, with a certain satisfaction, that there is practically nothing in the book which he wishes to withdraw, and concludes that the subject-matter of the book, "the classical quantum theory," is still necessary to an understanding of the newer developments, so that he can authorize the translation with a clear conscience. The reviewer gladly agrees with this verdict, with the added advantage that he may express himself more strongly. A proper knowledge of the classical quantum theory, as expounded here, is still as essential as ever, though, of course, the old theory is already simplified by the new theory by being shorn of some needless excrescences. But besides this nowhere else is to be found just that sequence of theorems and examples in generalized Newtonian dynamics which is relevant to atomic theory, old or new. Most of the necessary matter is accessible enough in dynamical treatises, but in these it must be searched for. Prof. Born has assembled it all ready to hand in his book.

Prof. Born in his original preface calls his book Vol. I., to indicate that it is a first approximation to an exact atomic theory. The hope of an early appearance of Vol. II. was at that time extremely faint, but abundantly justified. It is inspiring to realize that in less than three years the theory of the atom has so developed that Prof. Born may now quite properly be contemplating the drafting of Vol. II., a second but of course not necessarily a final approximation to atomic theory. R. H. FOWLER.

The Logic of Modern Physics. By P. W. BRIDGMAN. Pp. xvi+228. 10s. 6d. 1927. (New York: The Macmillan Co.)

Professor Bridgman is almost apologetic because, being primarily an experimenter, he ventures on criticism. But in truth no better credentials could be offered. Science is science, distinguished from mathematics and philosophy, because it is founded on experimental facts. Criticism of science is in grave danger of becoming as sterile, or at least as far divorced from its professed object, as criticism of art, because those who undertake it have no interest in, and no appreciation of, the facts which make their subject what it is. To this danger the corrective must come from those who, like Professor Bridgman, can both experiment and think.

We must expect of them an analysis of the problems rather than a solution of them, and this is what our author gives us. He starts from the doctrine that all the concepts of science are based on "operations," namely the experiments by means of which knowledge of them is derived, and that any proposition about them which cannot be interpreted in terms of these operations is simply meaningless. He then undertakes an examination of some of the chief concepts, and has no difficulty in showing that most scientific theory, and especially mathematical theory, judged by this criterion has no meaning whatever. He does not therefore propose to reject it; indeed it may rather be urged against him that he is inclined to accept too readily theories which may prove merely transitory; but he insists very strongly on the difficulties which they present.

If we understand Professor Bridgman rightly, he has no general solution to offer of the problem how propositions which have no experimental meaning may yet have scientific importance; he does not even discuss at any length the general solutions that others have offered. He deals with each theory on its own merits and tries to find particular solutions. It is quite possible that his method alone can succeed and that, in spite of his title, there is no "logic" of modern science, no set of general principles by means of which it can all be reduced to intellectual order. But there is one method of attack which it is rather strange that a writer of his proclivities and antecedents should have neglected; it is a more careful analysis than is usually given to the "operations" which give their meaning to experimental concepts. May there not be some feature common both to all these operations and to mathematical analysis which forms the bridge over the apparent gap between them? If there is, the discovery of it is most likely to come from the skilled experimenter of wide experience.

However, if nobody wrote about scientific principles who was not prepared to solve all difficulties, much valuable and entertaining literature would be lost. We are not sure that Professor Bridgman's book does much to solve the problems which are likely to interest the readers of the *Mathematical Gazette*, but we are sure that any solution which neglects the outlook which he describes with such vigour and clarity will not be of much value. N. R. C.

The Anatomy of Science. By G. N. Lewis. Pp. xii+219. 14s. net. 1926. (Yale Univ. Press.)

This book contains the lectures delivered under the Silliman foundation by Prof. Lewis, and surveys in a pleasant manner the whole field of science from arithmetic to biology, though, as might be expected, more than half the book deals with physics.

In the chapter on geometry little is said about the old non-Euclidean geometries of Lobatschefsky and Riemann, but a fuller account is given of two geometries in which translation plays the same rôle as in the Euclidean, but in which rotation has non-Euclidean properties. In the more important of these two, the circle of Euclid is replaced by a hyperbolic branch : this is, of course, the geometry of the restricted relativity theory, and was worked out in detail by the author and Prof. E. B. Wilson in 1912.

As everyone knows, the theory of light is at present a two-headed monster ; on the one hand, we have such phenomena as the Compton effect, which suggest a discontinuous quantum theory, and, on the other, the phenomena of interference, which seem to demand that light spreads out continuously when emitted from a source and is not kept tied up into discrete packets. Prof. Lewis has made an interesting attempt (*Nature*, 117, p. 236 (1926)) to bring interference into line with a quantum or corpuscular theory. In the relativity theory the interval between two points on a light-tract in space-time is zero ; the suggestion is that this vanishing is not merely a piece of mathematics but represents a physically real fact, that, *e.g.* when the eye sees a star (the interval between the events " the star emitting light " and " the eye seeing a star " being then zero) the eye is in virtual contact with the star, this virtual contact between the events being as real a contact as when the finger touches a table. It would seem difficult, however, to reorder events in space-time in accord with this suggestion, for it seems that A might be in contact with both B and C without B being in contact with C.

In the chapter on probability Prof. Lewis very clearly dissolves the well-known paradox as to how the equations of motion, which are themselves unchanged when t is changed to $-t$, can yet lead to sequences of phenomena apparently irreversible ; and he believes there is no evidence for the theory of the continual " degradation " of energy, once such a favourite corollary of the second law of thermodynamics. " All irreversible processes are mere shufflings and mixings with no element of novelty except that which arises from the vast number of elementary processes involved." The elementary processes themselves, even those involving radiation, are all reversible. (Compare Eddington, *The Internal Constitution of the Stars* (1926), p. 45 ff.)

Coming to questions of life and mind the author suggests that living things are cheats in the game of physics and chemistry and, like Maxwell's demon, may take advantage of the local fluctuations from the average state, and that it will never be possible to explain vital phenomena by physics and chemistry alone. On the contrary, the choice which is possible even to the lowest organisms is a departure from complete determinism. He does not, however, suggest as Weyl and Born do—though he almost does so—that even in atomic processes there is not complete determinism.

Matter and Gravity in Newton's Physical Philosophy. By A. J. Snow. Pp. 256. 7s. 6d. 1926. (Oxford Univ. Press.)

It must have been remarked by many that in the Newton volume of our Association there was one big omission : " Newton as a Theologian ". The book under review—if we except his speculations on Daniel, the Revelation and the three Witnesses—gives an adequate account of this side of Newton's work and deals with its relations to the contemporary philosophies. It seems

that though Newton thought that metaphysical truths could be deduced from scientific propositions only after *all* scientific facts were known, yet he was influenced in rejecting Descartes' physics not only by scientific reasons, but also because he thought its tendency was antitheistic, and he valued his own work because of its possible use in proving God's existence. "When I wrote my treatise [The *Principia*] about our system," he says in a letter to Bentley, "I had an eye upon such principles as might work with considering men for the belief of a Deity, and nothing can rejoice me more than to find it useful for that purpose." Nevertheless, he regarded the law of gravitation, quite in the modern way, as a purely descriptive law which was capable of proof, and he was misunderstood by Huyghens and Leibnitz because they never took that standpoint but persisted in regarding "attraction" in a metaphysical manner.

As to the cause of gravitation, Newton gave diffidently and almost under protest two suggestions. In the first, of a physical nature, he postulated an aether which became denser as the distance from solid bodies increased, and imagined, in a way not very clear, that the extra pressure in the denser parts would tend to drive a planet into the rarer portion nearer the sun. This aether, as the speculation developed, was suggested as the underlying cause of electricity, of fermentation, of the shining of the sun, and of animal motion. In the second, metaphysical, suggestion he apparently replaced this material or quasi-material aether by secret active principles which were the fundamental cause of all natural phenomena.

On scientific matters the author adopts the terminology and atmosphere of the physics of Newton's time, and this is well; for the use of modern terms like "kinetic energy" would make the difficulties Newton felt about the conservation of "motion" seem less real. But the author has misunderstood Newton in at least two places. He makes Newton say (p. 108), quoting *Opticks*, Qu. 22, that the aether was denser than quicksilver or gold, whereas Newton says the exact contrary and guesses it as being 700,000 times rarer than air; and he apparently thinks that Newton imagined sound was propagated by the aether (p. 107), misunderstanding the remark on sound in *Opticks*, Qu. 21.

There is an annotated bibliography at the end of the book. H. G. F.

Nicomachus of Gerasa: Introduction to Arithmetic. Translated into English by M. L. D'Ooge, with Studies in Greek Arithmetic by F. E. Robbins and L. C. Karpinski. University of Michigan Studies, Humanistic Series XVI. Pp. 318. 15s. 1926. (Macmillan.)

Arithmetic, in the eyes of the ancient Greek mathematicians, was primarily a philosophical study having no necessary connection with practical affairs. Indeed the Greeks gave a separate term, λογιστική, to the arithmetic of business, of monetary problems, the calculation of areas and so forth. No exposition of this branch of the subject has survived the intervening ages. To the Greeks "Arithmetic" did not mean what it means to us, addition, multiplication, division, the manipulation of fractions, decimals and root extractions. It rather meant what we should call the elementary theory of numbers, including the solution of indeterminate equations in rational numbers.

But little is known of the life of Nicomachus, who flourished about 100 A.D. and thus came under the influence of the Pythagorean philosophy. Judged by the standards of the mathematician, Nicomachus cannot rank with the leaders of the science even as it was known in antiquity; but estimated by the number of his commentators, translators and imitators, he is undoubtedly one of the most influential. From his own day until the sixteenth century there was hardly a place in the civilised world where he was not honoured as an arithmetician, or a time when his work was not regarded as the basis of the science. A series of introductory chapters in the present large volume discusses fully such subjects as the sources of Greek mathematics, Greek arithmetic before Nicomachus, Greek arithmetical notation and the works and philosophy of Nicomachus. In fact everything possible has been done to bring the reader to the true spirit of mathematical thought in Nicomachus'

time, and to estimate the author's work in the true perspective of distance. Our only doubt is whether the scientific value of Nicomachus' book is sufficient to warrant all the labour expended in preparing so elaborate an edition of his works.

Nicomachus' *Introductio Arithmetica* covers much the same ground as part of Euclid's arithmetical books VII.-IX., but it is far inferior in scientific value. Whereas Euclid built up a deductive treatise as rigorous as his geometry, Nicomachus was content to state rules and illustrate them by particular numerical examples, the working of which occasionally led him to draw faulty general conclusions. His detailed and imperfect classification of even and odd numbers is of no importance now and of interest only in showing a stage in the early development of the science. The section on "perfect numbers" is evidently based on Euclid, but lacks Euclid's accuracy of reasoning. Triangular, square and polygonal numbers, to which considerable importance was attached by Greek mathematicians, are discussed at length and lead to the most interesting results obtained by Nicomachus; the identities $1^3=1$, $2^3=3+5$, $3^3=7+9+11$, $4^3=13+\ldots+19$, and so on. W. E. H. B.

New Methods in Exterior Ballistics. By F. R. Moulton. Pp. vi + 260. 20s. net. 1926. (University of Chicago Press; Cambridge Univ. Press.)

This book is the outcome of the author's work during the war, when he was in charge of the Ballistics Branch of the Ordnance Department of the United States Army. The new demands upon artillery and the changes in artillery practice necessitated by barrage, counter-battery, long-range and anti-aircraft fire, as the war progressed, raised corresponding ballistic problems. The author states, in his introduction, that, on entering the United States Army, he took up the subject anew as a scientific problem requiring close coordination of adequate theory and well-conducted experiment. The object of this book, most of which was written in April and May, 1918, appears to be to supply the adequate theory.

In the first chapter the differential equations of motion of a heavy particle, fired from a rotating earth, are set out in full, account being taken of the convergence of verticals and the variation of gravity with height. The effects of the sun and moon on the particle are shown to be negligible. The theoretical and practical determination of gravity is treated in some detail in Chapter II, which also includes a discussion on the resistance of the air as a function of velocity and shape of head. A method of deducing the resistance and its velocity gradient from experimental observations is developed.

Chapter III is devoted to the general problem of solving differential equations numerically, and a finite-difference method is indicated for the solution of the equations of motion of a projectile. In Chapter IV are given two methods for calculating differential variations from normal conditions. The second method, due to Professor G. A. Bliss, is elegant and comparatively easy to apply. The results of such calculations are given in useful form in terms of weighting factors for a series of horizontal strata through which the trajectory passes. The validity of the numerical solutions is proved in Chapter V.

The three-dimensional motion of a rotating projectile is dealt with in Chapter VI. A solution is given for a small interval of time of flight and a numerical method of calculating the motion is indicated. The stability of the projectile is discussed at some length, the effects of the damping forces are dealt with, the phenomenon of drift is explained, and a sound method is given of deducing, from experimental observations, the various forces and couples exerted by the air.

The book is on the whole extremely interesting, but it seemed, on first reading, that some parts are unduly laboured. For example, in Chapter II eight pages are devoted to the theoretical and practical determination of gravity; the inclusion of this treatment is hardly consistent with the title of the book. Again, twelve pages are devoted, in Chapter VI, to proving Euler's equations; we think they might have been quoted without proof, since they are given in all books on rigid dynamics.

The proof, in Chapter V, of the validity of the numerical methods of solving differential equations is elegant and rigorous. This proof should be of con-

siderable interest to mathematicians generally; the need of it is apparently not generally realised by ballisticians.

The general development of the subject of ballistics during the war followed similar lines in most of the countries engaged; that in America naturally started somewhat late, but, from the assistance and information obtained from the allies, by whom a large proportion of the spade-work had been done, the development, once started, was very rapid. We think Professor Moulton's book would have been more interesting if, at various stages, he had compared his developments with those made in other countries.

There is much in this book which is of interest to mathematicians, while the whole forms an excellent example of the practical application of mathematical processes; to ballisticians it is indispensable as it contains many valuable contributions to the theory of external ballistics. F. R. W. HUNT.

Elementary Solid Geometry. By V. B. NAIK and S. B. BONDALE. Pp. vi+220. 1926. (Aryabhushan Press, Poona.)

The scope of the book can perhaps best be indicated by the fact that it is used by Bombay University as a course for Intermediate Examinations.

The bookwork is well explained, but is rather too spun out to be read in its entirety. The chief interest lies in the collection of excellent examples, which are very numerous and of varying degrees of difficulty. The value of the collection would be enhanced if answers were given to such problems as admit of a definite answer. In a few cases this has been done, the required answer being given in the question itself, but this is quite exceptional.

Chapters I and II are introductory, and well written.

Chapter III is not well arranged, postulates and definitions and deductions from them being intermingled in a worrying manner.

Chapters IV, VI, and VIII deal with planes in relation to straight lines and other planes, skew lines, and projection.

Chapter V deals with surfaces of revolution, specially cone, cylinder, and sphere, and with tangent lines and planes.

Chapters VII and IX deal with polyhedra, specially tetrahedra, finishing with Euler's Theorem on the regular solids.

Chapter X is devoted to Mensuration of Solids, chiefly the simple ones. Simpson's Rule is mentioned on p. 186, but is hardly given the prominence it deserves.

Every chapter has an excellent set of examples attached to it. A. LODGE.

A Concise Geometrical Conics. By C. V. DURELL. Pp. xvi+100. 4s. 1927. (Macmillan.)

The range of this account of the elementary metrical properties of the conic extends to theorems on curvature, the spirit is described frankly in the Preface:

"This forms a companion volume to the author's *Projective Geometry*. It is designed primarily for those whose main study of the geometry of the conic is conducted along projective lines. A course of this character requires supplementing, since many metrical properties are best tackled without recourse to projection. But this auxiliary treatment should be as concise as possible.

"It is hoped, however, that the volume may also be of service to those who have not the time to study projective methods. Alternative methods of proof of certain theorems, best treated by projection, have therefore been added in an appendix, in order to make the volume self-contained."

The book illustrates that even a skilful and practised writer needs some liking for his material and some sympathy with his readers. Doubtless students to whom the whole subject is an unpleasant but necessary supplement to a projective course will be grateful for being told just how little they can get along with, and if they find nothing to arouse enthusiasm, that is no less than they will expect. But students of the other kind of which Mr. Durell speaks, who are numerous in the newer universities if rare in the public schools, deserve a book in which a systematic metrical development is made as attractive and credible as possible.

The style in which individual proofs are set out leaves nothing to be desired, except indeed by those of us who challenge the educational value in elementary work of extending automatically to negative squares results proved by methods of pure geometry applicable only to positive squares. All of the diagrams are well reproduced, and most of them are clear, but I can not understand why Mr. Durell has been content with some that are grossly inaccurate: in a normal class there are generally two or three boys who take a delight in elegant draughtsmanship, and surely a popular master would have had enthusiastic volunteers if he had called for them. A figure involving a hyperbola needs careful planning, and sometimes distortion seems almost the only way to avoid a muddle; also accuracy in the figure has nothing to do with logic in the proof. But when a figure is actually said to show the form of the hyperbola, we may reasonably expect something less like a pair of semicircles than we find on p. 9, and distortion in any case should hardly reach the point of two figures on pp. 78 and 79, where the two branches palpably have common tangents. Mention must be made of the exercises, which are numerous and for the most part straightforward; a volume of hints and solutions has been prepared.

The book is concise without being obscure, and although I do not think it will serve its secondary purpose, it will serve its primary purpose admirably.

E. H. N.

An Introduction to the Calculus. By C. V. Durell and R. M. Wright. Pp. 91. 2s. 6d. 1926. (G. Bell and Sons.)

This little book is in substance a reprint of certain chapters in the *Elementary Algebra* by the same authors, but there is no essential dependence on the parent work. The actual amount of bookwork is relatively slight, but a large field is covered by the graded sequences of examples. The treatment is based on the consideration of the gradients of graphs, leading to differentiation, and its inverse, integration. The result is an attractive and not misleading introduction to the subject.

T. A. A. B.

Stories about Mathematics-Land. By D. Ponton. Book I. Pp. 186. 3s. 6d. 1926. Book II. Pp. 160. 3s. 6d. 1927. (Dent.)

Exercises in Stories about Mathematics-Land. By D. Ponton. Book I. Pp. 77. 1s. 1927. (Dent.)

These "Stories" are similar to those in the author's *Stories about Numberland* recently reviewed in these columns. They serve up mathematics in an interesting way by means of a series of story lessons based on the activities of imaginary but attractive "figures," such as Gnome Knowall, The Twins, The Impatient Man, The Little Professor, Mr. Quadrilateral.

Book I begins with a story "On the numbers one to four," and carries on the development of the subject through the first four rules, factors, decimals, to the metric system.

Book II deals with Geometry, introducing the common geometrical shapes and various practical constructions, and with Algebra to Simultaneous Equations and Square Root. The "Story" method is not pursued beyond the first 50 pages, and the presentation becomes more ordinary, the treatment of the Algebra being, in fact, hardly modern.

Both these "Story" books are very suitable for use with young children, and the Exercises ("Work for Keen People") form a useful review and application of the subject-matter.

R. S. W.

Elementary Mathematics. Part II. By E. Sankey and A. Royds. Pp. 120. 1s. 6d. 1926. (G. Bell and Sons.)

This book of 120 pages consists of sixty-six exercises, consisting almost entirely of more or less practical questions ranging from the rudiments of Arithmetic to Elementary Mensuration and Solid Geometry. It is not easy to see in what respect it differs from many similar works: the exercises of course are quite useful to any teacher.

Progressive Arithmetical Tests. By H. McKay. Without Answers. 3rd, 4th and 5th years, 4½d.; 6th year, 6½d. Each year, 40 pp. 1926. (Oxford University Press.)

These little books are clearly printed, and consist of questions carefully graded. Many of these are illustrated with good and clear diagrams, which should interest the pupils, especially the younger: and incidentally the practical character both of the questions themselves and the illustrations will, when discussed by an interested teacher, add much to the general knowledge of his class.

Arithmetic. Parts III and IV. By C. V. Durell and R. C. Fawdry. Pp. 214-369. 1926. (G. Bell and Sons.)

This book deals with some important points which are very often neglected, or at least scantily treated in the textbooks. A good chapter on Graphs begins the book, and variations comes in almost necessary logical sequence. The meaning and consideration of "significant figures" is treated carefully and at length, with many excellent examples taken from every-day experience and from real statistical statements. On page 175 are ten extremely good test questions of a more abstract kind. The treatment of square root extraction is as usual inadequately explained: this process is really a very beautiful one, and has valuable extensions in the theory of equations. The triple nature of the process, namely, the selection of approximating squares, their economic calculation, and the relation between the number, the remainder, and the part of the root found ought to be carefully explained in all arithmetical books which treat the subject at all.

Much space is given to commercial questions (stocks, shares, profit and loss), which are well illustrated by graphs; there is a certain amount of trigonometry and a good deal about logarithms that might be more fitly included in another work.

An excellent feature in the book is the large collection of interesting and practical problems.

Fundamental Arithmetic. By P. B. Ballard. Book II. Pupil, pp. 66. 1s.; Teacher, pp. 48. 2s. Book III. Pupil, pp. 82. 1s. 2d.; Teacher, pp. 64. 2s. 3d. Book IV. Pupil, pp. 82. 1s. 2d.; Teacher, pp. 79. 2s. 3d. 1926. (University of London Press).

Each of these books contains two divisions, one for the teacher, the other for the pupil. Book II. (teacher's) has some wise remarks about textbooks, marking, brevity of sums, etc. Not all teachers will accept the dictum (p. 21) about multiplication, but the author does not expect universal acquiescence. The ideas of page 22 are sound but well known. The advantages of checking are emphasised. The pupil's volume has clear and simple sums with good and even amusing illustrations. In Book III. (teacher's) some sound remarks on long sums, on a very old institution of late somewhat ignored, namely, self marking, and on proportion, make interesting reading. The pupil's volume is similar to that corresponding to it in Book II.

Book IV. carries the work on and introduces occasional algebraic symbols. The notes for teachers are again interesting and sound.

There is little or no discussion of such interesting abstractions as square root and circulating decimals.

Elementary Arithmetic. By W. G. Borchardt. Pp. xii + 259 + xxvi. 3s. 1926. (Rivingtons.)

This book of 260 pages consists of a well-graduated set of exercises. At the beginning they deal with numbers up to 9, up to 30, up to 100, up to 1000, etc. This seems a good plan with very young children. The utmost range is to square and cube root by factors and to simple interest. As a collection of simple questions the book is useful, like many similar works already published.

J. E. A. Steggall.

Algebra to the Quadratic. By R. W. M. Gibbs. Pp. 140+160. 3s. net. 1927. (Oxford University Press.)

This book seems to me an admirable introduction to Algebra, and a boy who has mastered its contents will have a good foundation on which to build. The book is divided into two parts; the first contains the "bookwork," including worked out examples for illustration, and the second the exercises. There are exercises on every section of the bookwork, and generally the exercises for each section are divided into two parts, "A" and "B" exercises. By having the exercises separate from the rest a boy is unable to do them by merely looking at the explanation on the opposite page, while the "A" exercises are so arranged with copious hints that the average boy should both be able to work through them without difficulty, and at the same time in so doing gain a good grasp of the theory of the section concerned. I am glad to see that the author makes abundant use of checks wherever possible, and frequently appeals to arithmetic for explanation and illustration. The book is largely influenced by the Report of the Mathematical Association on "The Teaching of Mathematics in Preparatory Schools"; and negative quantities are not introduced till the second chapter. The author states in the preface that his aim has been "to write the bookwork so simply that young boys may have it set to them for self-preparation," but I am doubtful as to his success except in the case of clever boys: the explanation of some of the sections seems too brief for this purpose, though in the hands of the teacher who would amplify where necessary, this would be no deficiency. It would have been a help if the number of the page to which they refer had been put at the top of each section of the answers. J. W. H.

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467. Robert Dodsley, printer and publisher, was the eldest son of a schoolmaster, "a good mathematician and a careful educationist," and something of a poet.

468. The inverse word to development should be *involvement*, so that while we say that $1+x+x^2+\ldots$ is the development of $1\div(1-x)$, we should also say that the latter is the involvement of the former. But the latter term is never used, which frequently gives rise to a circumlocutory and sometimes defective mode of description.—De Morgan, "Development," *Penny Cyclopaedia.*

469. All things began in order, so shall they end, and so shall they begin again; according to the ordainer of order and mystical Mathematicks of the City of Heaven.—Sir Thomas Brown, *The Garden of Cyrus*, c. v.

470. Turn-up Diagrams. De Morgan's volume of *Tracts*, dated 1721-1740, contains Samuel Cunn's Appendix to the English translation of Commandine's *Euclid.*

Under date Jan. 1856, De Morgan notes: Cunn's Appendix is full of *turn-up* diagrams for the solid figures.

471.

> . . . as their nurses dandle them
> They crow binomial theorem,
> With views (it seems absurd to us)
> On differential calculus.
>
> —*My Dream*, "Bab Ballads."

472. "Honest as the day?" Do you know that only four days, out of three hundred and sixty-five, are honest?

On the four solstices the time of day agrees absolutely with the sun. And on not one other day of them all.—*The Amateur Inn*, A. P. Terhune [per Mr. G. N. Bates].

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Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

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(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.